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A BRST charge for non-critical $\mathcal{W}_{2,s}$ strings

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ABSTRACT

We present a general argument for the construction of BRST charges of the ‘non-critical’ $\mathcal{W}_{2,4}$, $\mathcal{W}_{2,5}$, $\mathcal{W}_{2,6}$, and $\mathcal{W}_{2,8}$ strings. This evidences the existence of BRST charges for a kind of soft-type algebras which can be constructed from two copies of quantum $\mathcal{W}_{2,s}$ algebras, ($s=3,4,5,6,8$).

1. Introduction

Much work has been done on the construction and analysis of the BRST charges for \mathcal{W} algebras since the construction of the nilpotent BRST charge for the \mathcal{W}_3 algebra by Thierry-Mieg [1]. An interesting BRST operator is the one associated with a direct product of two quantum \mathcal{W}_3 algebras [2]. This direct product does not form a \mathcal{W}_3 algebra but a soft-type algebra. This result can also be interpreted as the BRST charge for the \mathcal{W}_3 matter coupled to \mathcal{W}_3 -gravity, extending in this way the theory of two-dimensional gravity. In this construction, one copy of the \mathcal{W}_3 algebra corresponds to the matter system and the other to the \mathcal{W}_3 gravity sector that can be represented by an SL3 Toda field theory. It has also been proved that this BRST charge appears in the quantisation of some classical soft-type algebras [3]. Due to the non-linear properties of the \mathcal{W} algebra, this BRST charge provides a basis for study of a nontrivial extension of the cohomology of the \mathcal{W}_3 algebra.

Recently, some nilpotent BRST operators have been found with no reference as to whether the quantum algebra exists or not [4, 5]. In this construction, it is possible to obtain several nilpotent BRST operators using spin-2 generators and a spin- s generator formed from a free field and ghost fields. These BRST charges are usually referred to as the BRST operators of the $\mathcal{W}_{2,s}$ strings. This construction gives the BRST charge as the sum of two commuting BRST operators. Using certain indirect methods, it has been possible to relate these BRST operators to some quantum algebras, for instance, the two BRST charges for the case $s = 4$ are associated with quantum \mathcal{WB}_2 algebra, two of the four $s = 6$ BRST operators can be associated with quantum \mathcal{WG}_2 algebra [6]. The cohomology related to these nilpotent operators has been discussed in [5, 6].

In the case $s = 5$, it seems that there would appear to be only one nilpotent BRST operator and the nilpotency condition requires a value of the central charge of $c_t = 268$. On the other hand, a quantum $\mathcal{W}_{2,5}$ algebra that could be related to this BRST charge is consistent for a discrete set of values of the central charge that does not include 268. A question that can be raised is what kind of algebraic

structure is behind the quantum BRST operator constructed in [4].

In the case of the $\mathcal{W}_{2,5}$ algebra, we should notice that the value of the central charge ($c_t = 268$), required by the nilpotency of the BRST operator, is not included in the discrete set of values for which a quantum $\mathcal{W}_{2,5}$ algebra is consistent. However, an interesting fact is that there are two particular values of the central charge, $c_{\pm} = 268/2 \pm 60\sqrt{5}$, for which the quantum $\mathcal{W}_{2,5}$ algebra is consistent. Furthermore, for these values of the central charge, the Jacobi identities are satisfied in a consistent way. Then, in an attempt to identify an underlying algebraic structure of the $s = 5$ BRST operator and using this apparent coincidence, one is tempted to consider the possibility of constructing a BRST charge associated with two copies of the quantum $\mathcal{W}_{2,5}$ algebras in similar fashion as has been done for the \mathcal{W}_3 algebra [2].

Here, we present the BRST operator of some classical non-linear algebras and the explicit expression for a BRST charge associated with $\mathcal{W}_{2,4} \times \mathcal{W}_{2,4}$. The left and right sectors are consistent quantum algebras at $c_{\pm} = \frac{1}{2}c_t \pm 60\sqrt{2}$, where $c_t = 4(3s^2 - 3s + 7)$ is the contribution of the ghost system $\{b, c\}$ and $\{\beta, \gamma\}$ with conformal weights $(2, -1)$ and $(s, 1 - s)$ respectively. Similar irrational values of the central charge occur in the quantum $\mathcal{W}_{2,s}$ algebras ($s = 5, 6, 8$). Then, a similar construction might be possible for these cases. This will be presented elsewhere.

2. BRST charges for some classical algebras.

First, we shall consider the BRST operator associated with the following classical algebras written in an OPE formalism

$$\begin{aligned} T(z)T(w) &= \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} \\ T(z)W(w) &= \frac{sW(w)}{(z-w)^2} + \frac{W'(w)}{z-w} \\ W(z)W(w) &= \frac{2\kappa^2 T^{s-1}(w)}{(z-w)^2} + \frac{\kappa^2 \partial T^{s-1}(w)}{z-w} \end{aligned} \tag{2.1}$$

The generators T and W have spin two and three respectively. Starting with a general Ansätze for the classical BRST charge and requiring the nilpotency of this operator one obtains

$$Q = Q_0 + Q_1 + Q_2 \quad (2.2)$$

where

$$Q_0(s) = \oint c(T + c'b + s\gamma'\beta + (s-1)\gamma'\beta) \quad (2.3)$$

$$Q_1(s) = \oint \gamma W \quad (2.4)$$

$$Q_2(s) = \oint \kappa^2 \gamma \gamma' T^{s-2} b \quad (2.5)$$

We have introduced the ghost-antighost systems (c, b) and (γ, β) of spin $(-1, 2)$ and $(1-s, s)$ associated with the generators T and W respectively. We should notice that (2.5) contains the field-dependent structure coefficients of the algebra.

Next, we consider two commuting copies of the algebra (2.1) and define the following generators

$$T = T_1 + T_2 \quad (2.6)$$

$$W = \frac{W_1}{\kappa_1} + i^{s-4} \frac{W_2}{\kappa_2} \quad (2.7)$$

Assuming the most general form for a BRST charge of the constraints (2.6) and (2.7), the existence of the algebras of the type (2.1) and requiring nilpotency for a BRST operator Q , we obtain

$$Q = Q_0 + Q_1 + Q_2 \quad (2.8)$$

where

$$Q_0(s, s) = \oint c(T + c'b + s\gamma'\beta + (s-1)\gamma\beta) \quad (2.9)$$

$$Q_1(s, s) = \oint \gamma W \quad (2.10)$$

$$Q_2(s, s) = \oint \gamma \gamma' (T_1^{s-2} - T_1^{s-3} T_2 + \dots - T_2^{s-2}) b \quad (2.11)$$

On the other hand, the soft-type algebra that arises from the two copies of (2.1) is the following

$$\begin{aligned} T(z)T(w) &= \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} \\ T(z)W(w) &= \frac{sW(w)}{(z-w)^2} + \frac{W'(w)}{z-w} \\ W(z)W(w) &= \frac{2F(T_1, T_2)T}{(z-w)^2} + \frac{\partial F(T_1, T_2)T}{z-w} \end{aligned} \quad (2.12)$$

where $F(T_1, T_2) = T_1^{s-2} - T_1^{s-3} T_2 + \dots - T_2^{s-2}$. The BRST charge for this non-linear algebra is also given by (2.2), and it can be obtained following the standard method for the construction of a classical BRST charge.

We thus see that at the classical level it was possible to associate a BRST charge with two copies of classical non-linear algebras after only requiring nilpotency to a general Ansatz of a BRST operator that contains generators of a non-linear classical algebra.

We would like to extend this straightforward method for the construction of a nilpotent BRST operator that will contain generators of consistent quantum algebras at certain values of the central charge, as to compute Q^2 we will need the OPE's of the generators within the general Ansatz.

In the next sections, we shall present a concrete example of this idea, the result can be seen as the quantisation of the kind of classical BRST operators presented in this section.

3. The quantum $\mathcal{W}_{2,s}$ algebras.

It is well known by now that the quantum $\mathcal{W}_{2,s}$ ($s = 3, 4, 5, 6, 7, 8$) algebras have the general form [7, 10, 11]

$$\begin{aligned} T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} \\ T(z)W(w) &= \frac{sW(w)}{(z-w)^2} + \frac{W'(w)}{z-w} \\ W(z)W(w) &= (c/s)[\mathbf{1}] + C_{ww}^w[W] \end{aligned} \tag{3.1}$$

where $T(z)$ is the spin-2 field generating a Virasoro algebra with central charge c and $W(z)$ is a spin- s primary field. When the spin- s is an odd number, the coefficient C_{ww}^w must be equal to zero. Several results have shown that the algebras in some cases are consistent for generic values of the central charge but in other cases the algebras are consistent only for a discrete set of values of the central charge [7, 10, 11]. For example, $\mathcal{W}_{2,4}$ is consistent for $c = 86 \pm 60\sqrt{2}$ with $C_{ww}^w = 0$, and for generic values of c with $C_{ww}^w \neq 0$ (except some particular values of c that can be read off from the coefficients of the algebra) [7, 8, 9, 10, 11]. This algebra is related to the \mathcal{WB}_2 algebra. For $s = 5$, the $\mathcal{W}_{2,5}$ algebra is consistent only for a discrete set of values of the central charge, namely $c = 6/7, -350/11, -7$ and some curious values $c = 134 \pm 60\sqrt{5}$. The first set are rational c -values which can be obtained from the minimal series of $E(6)$ Casimir algebra, from which one can obtain the $\mathcal{W}_{2,5}$ as a contraction algebra at these rational values of the central charge. The second set contains irrational values of the central charge. These values would apparently not correspond to RCFT's based on the result that states that having a finite set of primary fields in the operator content of a CFT implies the rationality of the central charge [12]. From (3.1), we could use the quantum algebras with $s = 4, s = 5$ and $s = 6$ at the irrational values of the central charge and construct a BRST charge associated with two copies of these quantum algebras which are the consistent quantum versions of the algebras (2.1).

The details of OPE (3.1) for the quantum $\mathcal{W}_{2,4}$ algebra are given below

$$\begin{aligned}
W(z)W(w) = & \frac{c/4}{(z-w)^8} + \frac{2T(w)}{(z-w)^6} + \frac{T'(w)}{(z-w)^5} \\
& + \frac{1}{(z-w)^4}(b_1\Lambda + b_2W(w) + 3/10T^{(2)}(w)) \\
& + \frac{1}{(z-w)^3}(1/2b_1\Lambda' + 1/2b_2W'(w) + 1/15T^{(3)}(w)) \\
& + \frac{1}{(z-w)^2}(5/36b_1\Lambda^{(2)} + 5/36b_2W^{(2)}(w) + b_3P + b_4D \\
& \quad + b_5A(w) + 1/84T^{(4)}(w)) \\
& + \frac{1}{z-w}(1/36b_1\Lambda^{(3)} + 1/36b_2W^{(3)}(w) + 1/2b_3P' + 1/2b_4D' \\
& \quad + 1/2b_5A'(w) + 1/560T^{(5)}(w))
\end{aligned} \tag{3.2}$$

where the coefficients and the composite fields are

$$\Lambda = (TT) - 3/10T'' \tag{3.3}$$

$$P = (TT'') - (TT')' + 2/9(TT)'' - 1/42T^{(4)} \tag{3.4}$$

$$D = (T\Lambda) - 1/6\Lambda'' \tag{3.5}$$

$$A = (TW) - 1/6W'' \tag{3.6}$$

$$\begin{aligned}
b_1 &= \frac{42}{22+5c} \\
b_2 &= \sqrt{\frac{54(24+c)(196-172c+c^2)}{(-1+2c)(22+5c)(68+7c)}} \\
b_3 &= \frac{3(-524+19c)}{10(-1+2c)(68+7c)} \\
b_4 &= \frac{24(13+72c)}{(-1+2c)(22+5c)(68+7c)} \\
b_5 &= \frac{28b_2}{3(24+c)}
\end{aligned} \tag{3.7}$$

in this case we are considering the values of the central charge $c_{\pm} = 86 \pm 60\sqrt{2}$ for

which $b_2 = 0 = b_5$.

4. The quantum BRST charge

First, we can give some generalities of the BRST charge associated with a quantum $\mathcal{W}_{2,s}$ algebra (3.1). The most general form of the BRST charge can in principle be written as the sum of s BRST charges, $Q_n(s)$, each one with a ghost number and conformal weight equal to one and a $(\beta - \gamma)$ -ghost-number equal to n [14]

$$Q(s) = Q_0(s) + Q_1(s) + \dots + Q_{s-1}(s)$$

At the quantum level, the classical result $Q_n(s) = 0$ for $n \geq 3$ would appear to be preserved [13, 4, 14]. The nilpotency of this BRST charge is then equivalent to the following

$$Q_0^2 = 0 \tag{4.1}$$

$$[Q_0, Q_1] = 0 \tag{4.2}$$

$$[Q_0, Q_2] + \frac{1}{2}[Q_1, Q_1] = 0 \tag{4.3}$$

The characteristics of Q_0 and the condition (4.2) result in the following form for Q_0

$$Q_0 = \oint c(T + \frac{1}{2}T_{bc} + T_{\beta\gamma}) \tag{4.4}$$

$$T_{bc} = -2bc' - b'c \tag{4.5}$$

$$T_{\beta\gamma} = -s\beta\gamma' - (s-1)\beta'\gamma \tag{4.6}$$

Q_0 in (4.4) satisfies the nilpotency condition provided that the central charge is

given by

$$c = 4(3s^2 - 3s + 7) \quad (4.7)$$

The form of Q_1 is given by

$$Q_1 = \oint \gamma W(z) \quad (4.8)$$

Since $W(z)$ is a primary field, the condition $[Q_0, Q_1] = 0$ will automatically be satisfied and any possible extra term has to be equal to zero [14]. These results can also be applied to BRST operators associated with $\mathcal{W}_{2,s} \times \mathcal{W}_{2,s}$ algebras, but with the generators T and W defined similarly as in (2.6) and (2.7). Then

$$Q_1(4, 4) = \oint \gamma \left[\frac{W_1(z)}{\sqrt{b_{4+}/2}} + \frac{W_2(z)}{\sqrt{b_{4-}/2}} \right] \quad (4.9)$$

where $b_{4\pm}$ are fixed by the condition (4.3) and coincide with (3.7) evaluated at $c_{\pm} = 86 \pm 60\sqrt{2}$.

In a previous section, we have presented the general form for the BRST charge associated with classical algebras of the type (2.1) and (2.12). In particular, the term $Q_2(s, s)$ associated with two classical algebras (2.1) is given in (2.11). Here we present the quantum version for the case of (4, 4), that in principle is of the form

$$Q_2(s, s) = \oint \gamma \gamma' (T_1^{s-2} - T_1^{s-3} T_2 + \dots T_2^{s-2}) b + O(\hbar^k) \quad (4.10)$$

In the case of the BRST operator for $\mathcal{W}_{2,4} \times \mathcal{W}_{2,4}$, we have obtained that indeed

$Q_3 = 0$ and Q_2 is given as follows[†]

$$\begin{aligned}
Q_2(4,4) = & \oint \gamma \gamma' [(T_1 T_1) - T_1 T_2 + (T_2 T_2)] b \\
& + \gamma \gamma' b \left[\frac{-7}{4} (T_1 + T_2)'' + 2\sqrt{2} (T_1 - T_2)'' \right] \\
& + \gamma \gamma^{(3)} b \left[\frac{7}{4} (T_1 + T_2) - \frac{7\sqrt{2}}{2} (T_1 - T_2) \right] \\
& + \gamma \gamma^{(2)} b' \left[\frac{5}{2} (T_1 + T_2) - \frac{11\sqrt{2}}{2} (T_1 - T_2) \right] \\
& + \gamma \gamma' b^{(2)} \left[\frac{7}{4} (T_1 + T_2) - \frac{7\sqrt{2}}{2} (T_1 - T_2) \right] \\
& - \frac{7}{48} (45 \gamma \gamma^{(5)} b - 76 \gamma \gamma^{(3)} b^{(2)} - 45 \gamma \gamma^{(2)} b^{(3)}) \\
& - \frac{7}{4} (\gamma \gamma^{(2)} c' b b' + \gamma \gamma' c' b b^{(2)}) \\
& + \frac{7}{4} \gamma \gamma' \gamma^{(2)} b \beta'
\end{aligned} \tag{4.11}$$

We have started with a general Ansatz for Q_2 and imposed the condition (4.3) to fix the coefficients of the Ansatz. We should also notice that only the OPE WW (3.2) with arbitrary coefficients b_i is needed and these will be fixed by (4.3) and given by (3.7) evaluated at $c_{\pm} = 86 \pm 60\sqrt{2}$. Finally, a BRST charge associated with $\mathcal{W}_{2,4} \times \mathcal{W}_{2,4}$ is given by (4.4), (4.9) and (4.11).

5. Conclusions

These results have shown the existence of a BRST charge associated with an algebraic structure $\mathcal{W}_{2,4} \times \mathcal{W}_{2,4}$, which is of a different nature as the ones constructed in [5]. It would be interesting to see their connection. Our construction has been possible due to the existence of quantum non-linear algebras at particular irrational values of the central charges of the form $c = \frac{1}{2}c_t \pm \sqrt{k}$, where c_t is the value of the central charge needed to construct a nilpotent BRST charge. These

[†] We have used the mathematica program in [15]

values occur for the $\mathcal{W}_{2,s}$ ($s = 4, 5, 6, 8$) and it would be interesting to investigate further the meaning of these special values and in particular, the existence of similar BRST charges.

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